A Proof of Jensen's Inequality

Let $\alpha_1, \ldots, \alpha_n$ be real numbers with $\alpha_1 + \cdots + \alpha_n = 1$. We consider the difference

$$\Delta = \sum_{k=1}^{n} \alpha_k x_k^2 - \left(\sum_{k=1}^{n} \alpha_k x_k \right)^2 .$$

Since $\sum_{j=1}^{n} \alpha_j = 1$ we can write

$$\sum_{k=1}^{n} \alpha_k x_k^2 = \sum_{k,j=1}^{n} \alpha_k \alpha_j x_k^2 \,,$$

hence

$$\Delta = \sum_{k,j=1}^{n} \alpha_k \alpha_j \left(x_k^2 - x_k x_j \right) = \sum_{k \neq j}^{n} \alpha_k \alpha_j \left(x_k^2 - x_k x_j \right)$$

$$= \sum_{k<j}^{n} \alpha_k \alpha_j \left(x_k^2 - x_k x_j + x_j^2 - x_j x_k \right) = \sum_{k<j}^{n} \alpha_k \alpha_j \left(x_k - x_j \right)^2 .$$

For positive weights α_k, we obtain Jensen's inequality.